

Contact processes with random vertex weights on oriented lattices

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Abstract:

In this paper we are concerned with contact processes with random vertex weights on oriented lattices. In our model, we assume that each vertex x of Z^d takes i. i. d. positive random value $\rho(x)$. Vertex y infects vertex x at rate proportional to $\rho(x)\rho(y)$ when and only when there is an oriented edge from y to x . We give the definition of the critical value λ_c of infection rate under the annealed measure and show that $\lambda_c = [1 + o(1)]/(dE\rho^2)$ as d grows to infinity. Classic contact processes on oriented lattices and contact processes on clusters of oriented site percolation are two special cases of our model.

Keywords: Contact process, random vertex weights, oriented lattice, critical value.

1 Introduction

In this paper we are concerned with contact processes with random vertex weights on oriented lattices. For d -dimensional oriented lattice Z^d , there is an oriented edge from x to $x + e_i$ for each $x \in Z^d$ and $1 \leq i \leq d$, where

$$e_i = (0, \dots, 0, \underset{i\text{th}}{1}, 0, \dots, 0).$$

For $x, y \in Z^d$, we denote by $x \rightarrow y$ when $y - x \in \{e_i\}_{1 \leq i \leq d}$. We denote by O the origin of Z^d .

Let ρ be a positive random variable such that $P(\rho > 0) > 0$ and $P(\rho \leq M) = 1$ for some $M > 0$. Let $\{\rho(x)\}_{x \in Z^d}$ be i. i. d. random variables such that $\rho(O)$ and ρ have the same distribution. When $\{\rho(x)\}_{x \in Z^d}$ is given, the contact process with random vertex weights on oriented lattice Z^d is a spin system with

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state space $\{0, 1\}^{Z^d}$ and flip rates function given by

$$c(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \sum_{y: y \rightarrow x} \rho(x) \rho(y) \eta(y) & \text{if } \eta(x) = 0 \end{cases} \quad (1.1)$$

for each $(x, \eta) \in Z^d \times \{0, 1\}^{Z^d}$, where $\lambda > 0$ is a positive parameter called the infection rate. More details of the definition of spin systems can be find in Chapter 3 of [9].

Intuitively, this contact process describes the spread of an infection disease. Vertices in state 0 are healthy and vertices in state 1 are infected. An infected vertex waits for an exponential time with rate one to become healthy. A healthy vertex x may be infected by an infected vertex y when and only when there is an oriented edge from y to x . The infection between y and x occurs at rate in proportional to $\rho(x)\rho(y)$.

Please note that the assumption $P(\rho < M) = 1$ ensures the existence of our process according to the basic theory constructed in [8].

The contact processes with random vertex weights is introduced by Peterson in [13] on finite complete graphs. He proves that the infection rate λ has a critical value $\lambda_c = \frac{1}{E\rho^2}$ such that the disease survives for a long time with high probability when $\lambda > \lambda_c$ or dies out quickly with high probability when $\lambda < \lambda_c$.

Recently, contact processes in random environments or random graphs is a popular topic. In [1], Chatterjee and Durrett show that contact processes on random graphs with power law degree distributions have critical value 0. This result disproves the guess in [10, 11] that the critical value is strictly positive according to a non-rigorous mean-field analysis. In [13], Peterson shows that contact processes with random vertex weights on complete graphs have critical value $\frac{1}{E\rho^2}$, which is consistent with the estimation given by the mean-field calculation. In [2, 16], Yao and Chen shows that complete convergence theorem holds for contact processes in a random environment on $Z^d \times Z^+$. The random environment they set includes the bond percolation model as a special case.

In our model, if ρ satisfies that $P(\rho = 1) = 1 - P(\rho = 0) = p$, then our model can be regarded as contact processes on clusters of oriented site percolation on Z^d . In [7], Kesten shows that site percolation on Z^d has critical probability $[1 + o(1)]/2d$. We are inspired a lot by this result.

2 Main result

Before giving our main results, we introduce some notations. We assume that the random variables $\{\rho(x)\}_{x \in Z^d}$ are defined on a probability space (Ω, \mathcal{F}, P) . We denote by E the expectation operator with respect to P .

For any $\omega \in \Omega$, we denote by P_λ^ω the probability measure of our contact process on oriented lattice Z^d with infection rate λ and vertex weights $\{\rho(x, \omega)\}_{x \in Z^d}$. The probability measure P_λ^ω is called the quenched measure. We denote by E_λ^ω the expectation operator with respect to P_λ^ω . We define

$$P_{\lambda,d}(\cdot) = E[P_\lambda^\omega(\cdot)],$$

which is called the annealed measure. We denote by $E_{\lambda,d}$ the expectation operator with respect to $P_{\lambda,d}$.

For any $t \geq 0$, we denote by η_t the configuration of our process at the moment t . In this paper, we mainly deal with the case that all the vertices are infected at $t = 0$. In later sections, if we need deal with the case that

$$A = \{x : \eta_0(x) = 1\} \neq Z^d,$$

then we will point out the initial infected set A and write η_t as η_t^A . When η_t is with no upper script, we refer to the case that

$$\{x : \eta_0(x) = 1\} = Z^d.$$

According to basic coupling of spin systems, it is easy to see that

$$P_{\lambda,d}(\eta_t(O) = 1) \leq P_{\lambda,d}(\eta_s(O) = 1)$$

for $t > s$ and

$$P_{\lambda_1,d}(\eta_t(O) = 1) \leq P_{\lambda_2,d}(\eta_t(O) = 1)$$

for $\lambda_1 < \lambda_2$. As a result, it is reasonable to define the following critical value of the infection rate.

$$\lambda_c(d) = \sup \left\{ \lambda : \lim_{t \rightarrow +\infty} P_{\lambda,d}(\eta_t(O) = 1) = 0 \right\}. \quad (2.1)$$

Please note that our process is symmetric under the annealed measure $P_{\lambda,d}$. So, $P_{\lambda,d}(\eta_t(x) = 1)$ does not rely on the choice of x . As a result, when $\lambda < \lambda_c(d)$,

$$\lim_{t \rightarrow +\infty} P_{\lambda,d}(\eta_t(x) = 1 \text{ for some } x \in A) = 0$$

for any finite $A \subseteq Z^d$ and hence η_t converges weakly to the configuration that all the vertices are healthy as t grows to infinity.

Our main result is the following limit theorem of $\lambda_c(d)$.

Theorem 2.1. *Assume that $P(\rho > 0) > 0$ and $P(\rho < M) = 1$ for some $M > 0$, then*

$$\lim_{d \rightarrow +\infty} d\lambda_c(d) = \frac{1}{E\rho^2}. \quad (2.2)$$

Theorem 2.1 shows that the critical value $\lambda_c(d)$ is approximately inversely proportional to the dimension d , the ratio of which is the reciprocal of the second moment of ρ .

When $\rho \equiv 1$, our process is the classic contact process on oriented lattice. In this case, we write $\lambda_c(d)$ as λ_d . When ρ satisfies that

$$P(\rho = 1) = 1 - P(\rho = 0) = p$$

for some $p \in (0, 1)$, our process is the contact process on clusters of oriented site percolation on Z^d . In this case, we write $\lambda_c(d)$ as $\lambda_c(d, \text{site}, p)$. There are two direct corollaries of Theorem 2.1.

Corollary 2.2.

$$\lim_{d \rightarrow +\infty} d\lambda_d = 1. \quad (2.3)$$

We say y is x 's neighbor when $y \rightarrow x$, then Corollary 2.2 shows that λ_d is approximately to the reciprocal of the number of neighbors. In [4, 6, 12], Holley, Liggett, Griffeath and Pemantle show that this conclusion holds for contact processes on non-oriented lattices and regular trees. In [14], Xue shows that the same conclusion holds for threshold one contact processes on lattices and regular trees.

Corollary 2.3. For $p \in (0, 1)$,

$$\lim_{d \rightarrow +\infty} dp\lambda_c(d, \text{site}, p) = 1. \quad (2.4)$$

Corollary 2.3 shows that $\lambda_c = [1 + o(1)]/(dp)$ as d grows to infinity for contact processes on clusters of oriented site percolation. In [15], Xue claims that the same conclusion holds for contact process on clusters of oriented bond percolation on Z^d .

Please note that the critical value $\lambda_c(d)$ we define is under the annealed measure $P_{\lambda, d}$. We can also define critical value $\lambda_c(\omega)$ under the quenched measure such that

$$\lambda_c(\omega) = \sup \left\{ \lambda : \forall x \in Z^d, \lim_{t \rightarrow +\infty} P_\lambda^\omega(\eta_t(x) = 1) = 0 \right\}$$

for any $\omega \in \Omega$. $\lambda_c(\omega)$ is a random variable. However, according to the ergodic theorem for i. i. d. random variables, it is easy to see that

$$P(\omega : \lambda_c(\omega) = \lambda_c(d)) = 1.$$

So we only need to deal with the critical value under the annealed measure.

The proof of Theorem 2.1 is divide into two sections. In Section 3, we will prove that

$$\liminf_{d \rightarrow +\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}.$$

In Section 4, we will prove that

$$\limsup_{d \rightarrow +\infty} d\lambda_c(d) \leq \frac{1}{E\rho^2}.$$

3 Lower bound

In this section we give a lower bound of $\lambda_c(d)$.

Lemma 3.1. *For each $d \geq 1$,*

$$\lambda_c(d) \geq \frac{1}{dE\rho^2}$$

and hence

$$\liminf_{d \rightarrow +\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}.$$

Proof. We use f_t to denote

$$E_{\lambda,d}[\rho(O)\eta_t(O)] = E[\rho(O,\omega)P_\lambda^\omega(\eta_t(O) = 1)].$$

According to the flip rates function of η_t given by (1.1), $\rho(O,\omega)$ and $P_\lambda^\omega(\eta_t(O) = 1)$ are positive correlated. Therefore,

$$f_t \geq E\rho(O,\omega)E[P_\lambda^\omega(\eta_t(O) = 1)] = E\rho P_{\lambda,d}(\eta_t(O) = 1).$$

Hence,

$$P_{\lambda,d}(\eta_t(O) = 1) \leq \frac{f_t}{E\rho}. \quad (3.1)$$

Please note that the assumption $P(\rho > 0) > 0$ ensures that $E\rho > 0$.

According to Hille-Yosida Theorem and (1.1),

$$\begin{aligned} \frac{d}{dt}P_\lambda^\omega(\eta_t(O) = 1) &= -P_\lambda^\omega(\eta_t(O) = 1) \\ &\quad + \lambda \sum_{y:y \rightarrow O} \rho(O)\rho(y)P_\lambda^\omega(\eta_t(O) = 0, \eta_t(y) = 1) \\ &\leq -P_\lambda^\omega(\eta_t(O) = 1) + \lambda \sum_{y:y \rightarrow O} \rho(O)\rho(y)P_\lambda^\omega(\eta_t(y) = 1). \end{aligned}$$

Therefore,

$$\frac{d}{dt}f_t \leq -f_t + \lambda \sum_{y:y \rightarrow O} E[\rho^2(O)\rho(y)P_\lambda^\omega(\eta_t(y) = 1)]. \quad (3.2)$$

For each y such that $y \rightarrow O$, $\eta_t(y)$ is only influenced by the vertices from which there are oriented pathes to y . Therefore, $\rho(O)$ is independent of $\rho(y)P_\lambda^\omega(\eta_t(y) = 1)$ and hence

$$E[\rho^2(O)\rho(y)P_\lambda^\omega(\eta_t(y) = 1)] = E\rho^2 P_{\lambda,d}(\eta_t(y) = 1) = E\rho^2 f_t. \quad (3.3)$$

By (3.2) and (3.3),

$$\frac{d}{dt}f_t \leq (d\lambda E\rho^2 - 1)f_t. \quad (3.4)$$

According to Greenwood inequality and (3.4),

$$f_t \leq f_0 \exp\{(d\lambda E\rho^2 - 1)t\}$$

and hence

$$\lim_{t \rightarrow +\infty} f_t = 0 \quad (3.5)$$

when $\lambda < \frac{1}{dE\rho^2}$.

Lemma 3.1 follows (3.1) and (3.5). \square

4 Upper bound

In this section we will prove that $\liminf_{d \rightarrow +\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}$.

First we define the contact process $\hat{\eta}_t$ where disease spreads through the opposite direction of the oriented edges. For any $\omega \in \Omega$, The flip rates of $\hat{\eta}_t$ with random vertex weights $\{\rho(x, \omega)\}_{x \in Z^d}$ is given by

$$\hat{c}(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \sum_{y: x \rightarrow y} \rho(x)\rho(y)\eta(y) & \text{if } \eta(x) = 0. \end{cases}$$

Hence, for $\hat{\eta}_t$, y may infect x when and only when there is an edge from x to y .

According to the graphic representation of contact processes introduced by Harris in [5]. There is a dual of η_t and $\hat{\eta}_t$ such that

$$P_\lambda^\omega(\eta_t(O) = 1) = P_\lambda^\omega(\hat{\eta}_t^O \neq \emptyset), \quad (4.1)$$

where $\hat{\eta}_t^O$ is $\hat{\eta}_t$ with that $\{x \in Z^d : \hat{\eta}_0(x) = 1\} = \{O\}$.

Please note that in (4.1) we utilize the identification of $\hat{\eta}_t^O$ with

$$\{x \in Z^d : \hat{\eta}_t^O(x) = 1\}.$$

Since $\{\rho(x)\}_{x \in Z^d}$ are i. i. d., the events $\{\eta_t^O \neq \emptyset\}$ and $\{\hat{\eta}_t^O \neq \emptyset\}$ have the same distribution under the annealed measure $P_{\lambda,d}$. Therefore, according to (4.1),

$$P_{\lambda,d}(\eta_t(O) = 1) = P_{\lambda,d}(\eta_t^O \neq \emptyset). \quad (4.2)$$

To control the size of η_t^O from below, we introduce a Markov process ζ_t with state space $\{-1, 0, 1\}^{Z^d}$. For given $\{\rho(x)\}_{x \in Z^d}$, ζ_t evolves as follows. For each $x \in Z^d$, if $\zeta(x) = -1$, then x is frozen in the state -1 forever. If $\zeta(x) = 1$, then the value of x waits for an exponential time with rate one to become -1 . If $\zeta(x) = 0$, then the value of x flips to 1 at rate

$$\lambda \sum_{y: y \rightarrow x} \rho(x) \rho(y) 1_{\{\zeta(y)=1\}}.$$

So for ζ_t , when an infected vertex becomes healthy, then it is removed (in the state -1) and will never be infected.

We use ζ_t^O to denote ζ_t with $\{x \in Z^d : \zeta_0(x) = 1\} = \{O\}$ and $\{x \in Z^d : \zeta_0(x) = -1\} = \emptyset$. According to the basic coupling of Markov processes, there is a coupling of η_t and ζ_t under quenched measure P_λ^ω such that

$$\eta_t^O \supseteq \{x \in Z^d : \zeta_t^O(x) = 1\} \quad (4.3)$$

for any $t > 0$.

We use C_t to denote $\{x \in Z^d : \zeta_t^O(x) = 1\}$. Then, by (4.2) and (4.3),

$$\lim_{t \rightarrow +\infty} P_{\lambda,d}(\eta_t(O) = 1) \geq P_{\lambda,d}(\forall t, C_t \neq \emptyset). \quad (4.4)$$

We give another description of $\{\forall t, C_t \neq \emptyset\}$. Let $\{T_x\}_{x \in Z^d}$ be i. i. d. exponential times with rate 1. For any $x \rightarrow y$, let U_{xy} be exponential time with rate $\lambda \rho(x) \rho(y)$. We assume that all these exponential times are independent. For

$$O = x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n = x,$$

if $U_{x_j x_{j+1}} \leq T_{x_j}$ for each $0 \leq j \leq n-1$, then we say that there is an infected path with length n from O to x , which is denoted by $O \Rightarrow_n x$.

In the sense of coupling,

$$\{O \Rightarrow_n x\} = \{\exists t, x \in C_t\}.$$

Let $I_n = \{x : O \Rightarrow_n x\}$ and L_n be the set of infected pathes with length n from O . $\{\forall t, C_t \neq \emptyset\}$ is equivalent to that there are infinite many vertices which have ever been infected. Therefore,

$$\{\forall t, C_t \neq \emptyset\} = \{\forall n, I_n \neq \emptyset\}.$$

and

$$\begin{aligned} P_{\lambda,d}(\forall t, C_t \neq \emptyset) &= \lim_{n \rightarrow +\infty} P_{\lambda,d}(I_n \neq \emptyset) \\ &= \lim_{n \rightarrow +\infty} P_{\lambda,d}(|L_n| > 0) \\ &\geq \limsup_{n \rightarrow +\infty} \frac{(E_{\lambda,d}|L_n|)^2}{E_{\lambda,d}|L_n|^2} \end{aligned} \quad (4.5)$$

according to Hölder inequality.

To calculate $E_{\lambda,d}|L_n|$ and $E_{\lambda,d}|L_n|^2$, we utilize the simple random walk S_n on oriented lattice Z^d with $S_0 = O$ and

$$P(S_{n+1} - S_n = e_i) = \frac{1}{d}$$

for $1 \leq i \leq d$. Let $\{\widehat{S}_n\}_{n=0}^{+\infty}$ be an independent copy of $\{S_n\}_{n=0}^{+\infty}$. We assume that $\{S_n\}_{n=0}^{+\infty}$ and $\{\widehat{S}_n\}_{n=0}^{+\infty}$ are defined on probability space $(\widetilde{\Omega}, \mathcal{G}, \widetilde{P})$ and are independent of $\{\rho(x)\}_{x \in Z^d}$, $\{T_x\}_{x \in Z^d}$ and $\{U_{xy}\}_{x \rightarrow y}$. We denote by \widetilde{E} the expectation operator with respect to \widetilde{P} .

For a given path $O \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n$,

$$P_\lambda^\omega(U_{x_j x_{j+1}} < T_{x_j}, \forall 0 \leq j \leq n-1) = \prod_{j=0}^{n-1} \left[\frac{\lambda \rho(x_j, \omega) \rho(x_{j+1}, \omega)}{1 + \lambda \rho(x_j, \omega) \rho(x_{j+1}, \omega)} \right].$$

and hence

$$P_{\lambda,d}(U_{x_j x_{j+1}} < T_{x_j}, \forall 0 \leq j \leq n-1) = E \prod_{j=0}^{n-1} \left[\frac{\lambda \rho(x_j) \rho(x_{j+1})}{1 + \lambda \rho(x_j) \rho(x_{j+1})} \right]. \quad (4.6)$$

It is obviously that the right hand side of (4.6) does not rely on the oriented path x_1, x_2, \dots, x_n we choose.

As a result,

$$E_{\lambda,d}|L_n| = d^n E \prod_{j=0}^{n-1} \left[\frac{\lambda \rho(S_j) \rho(S_{j+1})}{1 + \lambda \rho(S_j) \rho(S_{j+1})} \right] \quad (4.7)$$

for any given first n steps (S_0, S_1, \dots, S_n) of the simple random walk.

Please note that we write E not $\widetilde{E} \times E$ in the right hand side of (4.7). We mean that the right hand side of (4.7) is a random variable with respect to \mathcal{G} and is a constant with probability one.

To calculate $E_{\lambda,d}|L_n|^2$, we introduce the following notations.

$$\begin{aligned}
\tau_1 &= \inf\{n \geq 0 : S_n = \widehat{S}_n, S_{n+1} = \widehat{S}_{n+1}\}, \\
\sigma_1 &= \inf\{n > \tau_1 : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\}, \\
L_1 &= \sigma_1 - \tau_1 + 1, \\
\tau_2 &= \inf\{n > \sigma_1 : S_n = \widehat{S}_n, S_{n+1} = \widehat{S}_{n+1}\}, \\
\sigma_2 &= \inf\{n > \tau_2 : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\}, \\
L_2 &= \sigma_2 - \tau_2 + 1, \\
&\dots\dots\dots \\
\tau_k &= \inf\{n > \sigma_{k-1} : S_n = \widehat{S}_n, S_{n+1} = \widehat{S}_{n+1}\}, \\
\sigma_k &= \inf\{n > \tau_k : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\}, \\
L_k &= \sigma_k - \tau_k + 1, \\
&\dots\dots\dots \\
T &= \sup\{k : \tau_k < +\infty\}.
\end{aligned}$$

Please note that $P(T < +\infty) = 1$ for $d \geq 4$ according to the conclusion proven in [3] that $P(\exists n > 0, S_n = \widehat{S}_n) < 1$ for $d \geq 4$. Therefore, τ_k, σ_k, L_k are finite for $k \leq T$.

Furthermore, we define

$$\begin{aligned}
A_0 &= \{0 \leq n < \tau_1 : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\}, \\
A_1 &= \{\sigma_1 < n < \tau_2 : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\}, \\
&\dots\dots\dots \\
A_{T-1} &= \{\sigma_{T-1} < n < \tau_T : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\}, \\
A_T &= \{n > \sigma_T : S_n = \widehat{S}_n, S_{n+1} \neq \widehat{S}_{n+1}\}.
\end{aligned}$$

For $0 \leq i \leq T$, we use K_i to denote $|A_i|$.

After all this prepare work, we give a lemma which is crucial for us to give upper bound of $\lambda_c(d)$.

Lemma 4.1. *Assume that $P(\rho > 0) > 0$ and $P(\rho < M) = 1$. If λ makes*

$$\widetilde{E} \left[\frac{2^{T + \sum_{j=0}^T K_j} M^{6T+4} \sum_{j=0}^T K_j (1 + \lambda M^2)^{2 \sum_{j=1}^T L_j + 2 \sum_{j=0}^T K_j}}{\lambda^{\sum_{j=1}^T L_j - T} (E\rho^2)^{\sum_{j=1}^T L_j + 2T + 2 \sum_{j=0}^T K_j}} \right] < +\infty, \quad (4.8)$$

then

$$\lambda_c(d) \leq \lambda.$$

Proof. For each $x \rightarrow z_1$ and $y \rightarrow z_2$, we define $F(x, y; z_1, z_2)$ as

$$P_\lambda^\omega(U_{xz_1} \leq T_x, U_{yz_2} \leq T_y).$$

By direct calculation,

$$F(x, y; z_1, z_2) \begin{cases} = \frac{\lambda^2 \rho(x) \rho(y) \rho(z_1) \rho(z_2)}{[1 + \lambda \rho(x) \rho(z_1)][1 + \lambda \rho(y) \rho(z_2)]} & \text{if } x \neq y \text{ and } z_1 \neq z_2, \\ = \frac{\lambda^2 \rho(x) \rho(y) \rho^2(z_1)}{[1 + \lambda \rho(x) \rho(z_1)][1 + \lambda \rho(y) \rho(z_1)]} & \text{if } x \neq y \text{ and } z_1 = z_2, \\ = \frac{\lambda \rho(x) \rho(z_1)}{1 + \lambda \rho(x) \rho(z_1)} & \text{if } x = y \text{ and } z_1 = z_2, \\ \leq \frac{2\lambda^2 \rho^2(x) \rho(z_1) \rho(z_2)}{[1 + \lambda \rho(x) \rho(z_1)][1 + \lambda \rho(x) \rho(z_2)]} & \text{if } x = y \text{ and } z_1 \neq z_2. \end{cases} \quad (4.9)$$

We denote by P_n the set of all the oriented paths from O with length n , then

$$\begin{aligned} E_{\lambda, d} |L_n|^2 &= \sum_{\mathbf{x} \in P_n} \sum_{\mathbf{y} \in P_n} P_{\lambda, d}(\forall 0 \leq i \leq n-1, U_{x_i x_{i+1}} \leq T_{x_i}, U_{y_i y_{i+1}} \leq T_{y_i}) \\ &= \sum_{\mathbf{x} \in P_n} \sum_{\mathbf{y} \in P_n} E P_\lambda^\omega(\forall 0 \leq i \leq n-1, U_{x_i x_{i+1}} \leq T_{x_i}, U_{y_i y_{i+1}} \leq T_{y_i}) \\ &= \sum_{\mathbf{x} \in P_n} \sum_{\mathbf{y} \in P_n} E \left[\prod_{i=0}^{n-1} F(x_i, y_i; x_{i+1}, y_{i+1}) \right] \\ &= d^{2n} \sum_{\mathbf{x} \in P_n} \sum_{\mathbf{y} \in P_n} \frac{1}{d^{2n}} E \left[\prod_{i=0}^{n-1} F(x_i, y_i; x_{i+1}, y_{i+1}) \right] \\ &= d^{2n} (\tilde{E} \times E) \left[\prod_{i=0}^{n-1} F(S_i, \hat{S}_i; S_{i+1}, \hat{S}_{i+1}) \right]. \end{aligned}$$

Therefore, by (4.7),

$$\frac{(E_{\lambda, d} |L_n|)^2}{E_{\lambda, d} |L_n|^2} = \left\{ \tilde{E} \left[\frac{E \prod_{i=0}^{n-1} F(S_i, \hat{S}_i; S_{i+1}, \hat{S}_{i+1})}{\left(E \prod_{i=0}^{n-1} \frac{\lambda \rho(S_i) \rho(S_{i+1})}{1 + \lambda \rho(S_i) \rho(S_{i+1})} \right)^2} \right] \right\}^{-1}. \quad (4.10)$$

Then by (4.4) and (4.5), $\lambda \geq \lambda_c(d)$ when

$$\limsup_{n \rightarrow +\infty} \tilde{E} \left[\frac{E \prod_{i=0}^{n-1} F(S_i, \hat{S}_i; S_{i+1}, \hat{S}_{i+1})}{\left(E \prod_{i=0}^{n-1} \frac{\lambda \rho(S_i) \rho(S_{i+1})}{1 + \lambda \rho(S_i) \rho(S_{i+1})} \right)^2} \right] < +\infty.$$

Now we control

$$\tilde{E} \left[\frac{E \prod_{i=0}^{n-1} F(S_i, \hat{S}_i; S_{i+1}, \hat{S}_{i+1})}{\left(E \prod_{i=0}^{n-1} \frac{\lambda \rho(S_i) \rho(S_{i+1})}{1 + \lambda \rho(S_i) \rho(S_{i+1})} \right)^2} \right]$$

from above.

For the denominator $\left(E \prod_{i=0}^{n-1} \frac{\lambda \rho(S_i) \rho(S_{i+1})}{1 + \lambda \rho(S_i) \rho(S_{i+1})}\right)^2$, if $S_i = \widehat{S}_i$ or $S_{i+1} = \widehat{S}_{i+1}$, then we narrow the factor $\frac{1}{1 + \lambda \rho(S_i) \rho(S_{i+1})}$ to $\frac{1}{1 + \lambda M^2}$, where $P(\rho < M) = 1$ as we assumed.

For the numerator $E \prod_{i=0}^{n-1} F(S_i, \widehat{S}_i; S_{i+1}, \widehat{S}_{i+1})$, if $S_i = \widehat{S}_i$ or $S_{i+1} = \widehat{S}_{i+1}$, then we enlarge the factor $\frac{1}{1 + \lambda \rho(S_i) \rho(S_{i+1})}$ to 1. If $i \in A_k$ for some k , then by (4.9), we enlarge the factors

$$2\lambda^2 \rho^2(S_i) \rho(S_{i+1}) \rho(\widehat{S}_{i+1}) \text{ and } \lambda^2 \rho(S_{i-1}) \rho(\widehat{S}_{i-1}) \rho^2(S_i)$$

to

$$2\lambda^2 M^2 \rho(S_{i+1}) \rho(\widehat{S}_{i+1}) \text{ and } \lambda^2 \rho(S_{i-1}) \rho(\widehat{S}_{i-1}) M^2.$$

If $i = \tau_k$ for some k , then we enlarge the factors

$$\lambda^2 \rho(S_{i-1}) \rho(\widehat{S}_{i-1}) \rho^2(S_i) \text{ and } \lambda \rho(S_i) \rho(S_{i+1})$$

to

$$\lambda^2 \rho(S_{i-1}) \rho(\widehat{S}_{i-1}) M^2 \text{ and } \lambda M \rho(S_{i+1}).$$

If $i = \sigma_k$ for some k , then we enlarge the factors

$$\lambda \rho(S_{i-1}) \rho(S_i) \text{ and } 2\lambda^2 \rho^2(S_i) \rho(S_{i+1}) \rho(\widehat{S}_{i+1})$$

to

$$\lambda \rho(S_{i-1}) M \text{ and } 2\lambda^2 M^2 \rho(S_{i+1}) \rho(\widehat{S}_{i+1}).$$

After all these operations, we can cancel many common factors in the numerator and denominator. For example, if $i, j \in A_k$ and $l \notin A_k$ for each $i < l < j$, then we can abstract

$$\left[E \frac{\prod_{l=1}^{j-i-1} \rho_l^2}{\prod_{l=1}^{j-i-2} (1 + \lambda \rho_l \rho_{l+1})} \right]^2$$

from both numerator and denominator and cancel this common factor, where $\{\rho_l\}_{l=1}^{i-j-1}$ are i. i. d. and have the same distribution as that of ρ .

Therefore, after all the above operations, it is easy to see that

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \frac{E \prod_{i=0}^{n-1} F(S_i, \widehat{S}_i; S_{i+1}, \widehat{S}_{i+1})}{\left(E \prod_{i=0}^{n-1} \frac{\lambda \rho(S_i) \rho(S_{i+1})}{1 + \lambda \rho(S_i) \rho(S_{i+1})}\right)^2} \\ & \leq \frac{2^{T + \sum_{j=0}^T K_j} M^{6T+4} \sum_{j=0}^T K_j (1 + \lambda M^2)^{2 \sum_{j=1}^T L_j + 2 \sum_{j=0}^T K_j}}{\lambda^{\sum_{j=1}^T L_j - T} (E \rho^2)^{\sum_{j=1}^T L_j + 2T + 2 \sum_{j=0}^T K_j}} \end{aligned}$$

and

$$\begin{aligned}
& \limsup_{n \rightarrow +\infty} \tilde{E} \left[\frac{E \prod_{i=0}^{n-1} F(S_i, \hat{S}_i; S_{i+1}, \hat{S}_{i+1})}{\left(E \prod_{i=0}^{n-1} \frac{\lambda \rho(S_i) \rho(S_{i+1})}{1 + \lambda \rho(S_i) \rho(S_{i+1})} \right)^2} \right] \\
& \leq \tilde{E} \left[\frac{2^{T + \sum_{j=0}^T K_j} M^{6T+4} \sum_{j=0}^T K_j (1 + \lambda M^2)^{2 \sum_{j=1}^T L_j + 2 \sum_{j=0}^T K_j}}{\lambda^{\sum_{j=1}^T L_j - T} (E \rho^2)^{\sum_{j=1}^T L_j + 2T + 2 \sum_{j=0}^T K_j}} \right]. \tag{4.11}
\end{aligned}$$

Lemma 4.1 follows (4.4), (4.5), (4.10) and (4.11). \square

Finally, we give the proof of $\limsup_{n \rightarrow +\infty} d\lambda_c(d) \leq \frac{1}{E\rho^2}$.

Proof of $\limsup_{n \rightarrow +\infty} d\lambda_c(d) \leq \frac{1}{E\rho^2}$. Let

$$\tau = \inf\{n > 0 : S_n = \hat{S}_n\}.$$

Then according to (2.9) of [3],

$$P(2 \leq \tau < +\infty) \leq \frac{C_1}{d^2},$$

where C_1 does not depend on d . Therefore, according to strong Markov property,

$$\begin{aligned}
& P(T = m, K_i = k_i \text{ for } 0 \leq i \leq m, L_i = l_i \text{ for } 1 \leq i \leq m) \\
& \leq \left(\frac{C_1}{d^2} \right)^{\sum_{i=0}^m k_i + m - 1} \left(\frac{1}{d} \right)^{\sum_{i=1}^m l_i - m} \tag{4.12}
\end{aligned}$$

for all possible m, k_i, l_i . Please note that k_0 may take 0 but $l_i \geq 1$ and $k_i \geq 1$ for $1 \leq i \leq m$.

Let

$$\lambda = \frac{\gamma}{dE\rho^2}$$

for fixed $\gamma > 1$, then by (4.12),

$$\begin{aligned}
& \tilde{E} \left[\frac{2^{T+\sum_{j=0}^T K_j} M^{6T+4} \sum_{j=0}^T K_j (1+\lambda M^2)^2 \sum_{j=1}^T L_j+2 \sum_{j=0}^T K_j}{\sum_{j=1}^T L_j-T (E\rho^2)^{\sum_{j=1}^T L_j+2T+2} \sum_{j=0}^T K_j} \right] \\
& \leq \sum_{m=0}^{+\infty} \sum_{k_0=0}^{+\infty} \sum_{k_1=1}^{+\infty} \cdots \sum_{k_m=1}^{+\infty} \sum_{l_1=1}^{+\infty} \cdots \sum_{l_m=1}^{+\infty} \left(\frac{C_1}{d^2} \right)^{\sum_{i=0}^m k_i+m-1} \left(\frac{1}{d} \right)^{\sum_{i=1}^m l_i-m} \\
& \times \frac{2^{m+\sum_{j=0}^m k_j} M^{6m+4} \sum_{j=0}^m k_j (1+\lambda M^2)^2 \sum_{j=1}^m l_j+2 \sum_{j=0}^m k_j}{\lambda^{\sum_{j=1}^m l_j-m} (E\rho^2)^{\sum_{j=1}^m l_j+2m+2} \sum_{j=0}^m k_j}. \\
& = \sum_{m=0}^{+\infty} \sum_{k_0=0}^{+\infty} \left(\frac{2C_1 M^6 \lambda}{d(E\rho^2)^2} \right)^m \left[\frac{2C_1 M^4 (1+\lambda M^2)^2}{d^2 (E\rho^2)^2} \right]^{k_0} \\
& \times \left[\sum_{l=1}^{+\infty} \left(\frac{2C_1 M^4 (1+\lambda M^2)^2}{d^2 (E\rho^2)^2} \right)^l \right]^m \left[\sum_{l=1}^{+\infty} \left(\frac{(1+\lambda M^2)^2}{d\lambda E\rho^2} \right)^l \right]^m \frac{d^2}{C_1}. \tag{4.13}
\end{aligned}$$

Since $\lambda = \frac{\gamma}{dE\rho^2}$ for $\gamma > 1$,

$$\frac{2C_1 M^6 \lambda}{d(E\rho^2)^2} \leq \frac{C_2}{d^2}$$

and

$$\frac{2C_1 M^4 (1+\lambda M^2)^2}{d^2 (E\rho^2)^2} \leq \frac{C_3}{d^2}$$

for sufficiently large d , where C_2 and C_3 does not depend on d (but may depend on γ and ρ).

We choose $\hat{\gamma}$ such that $1 < \hat{\gamma} < \gamma$, then for sufficiently large d ,

$$\frac{(1+\lambda M^2)^2}{d\lambda E\rho^2} = \frac{\left(1 + \frac{\gamma M^2}{dE\rho^2}\right)^2}{\gamma} < \frac{1}{\hat{\gamma}}.$$

Then, by (4.13),

$$\begin{aligned}
& \tilde{E} \left[\frac{2^{T+\sum_{j=0}^T K_j} M^{6T+4} \sum_{j=0}^T K_j (1+\lambda M^2)^2 \sum_{j=1}^T L_j+2 \sum_{j=0}^T K_j}{\lambda^{\sum_{j=1}^T L_j-T} (E\rho^2)^{\sum_{j=1}^T L_j+2T+2} \sum_{j=0}^T K_j} \right] \\
& \leq \frac{d^2}{C_1} \sum_{m=0}^{+\infty} \sum_{k_0=0}^{+\infty} \left(\frac{C_2}{d^2} \right)^m \left(\frac{C_3}{d^2} \right)^{k_0} \left[\sum_{l=1}^{+\infty} \left(\frac{C_3}{d^2} \right)^l \right]^m \left[\sum_{l=1}^{+\infty} \left(\frac{1}{\hat{\gamma}} \right)^l \right]^m.
\end{aligned}$$

For sufficiently large d ,

$$\sum_{l=1}^{+\infty} \left(\frac{C_3}{d^2}\right)^l = \frac{C_3}{d^2 - C_3} \leq \frac{C_4}{d^2}$$

and

$$\sum_{k_0=0}^{+\infty} \left(\frac{C_3}{d^2}\right)^{k_0} = \frac{d^2}{d^2 - C_3} \leq 2,$$

where C_4 does not depend on d .

Therefore,

$$\begin{aligned} & \tilde{E} \left[\frac{2^{T + \sum_{j=0}^T K_j} M^{6T+4 \sum_{j=0}^T K_j} (1 + \lambda M^2)^{2 \sum_{j=1}^T L_j + 2 \sum_{j=0}^T K_j}}{\lambda^{\sum_{j=1}^T L_j - T} (E\rho^2)^{\sum_{j=1}^T L_j + 2T + 2 \sum_{j=0}^T K_j}} \right] \\ & \leq \frac{2d^2}{C_1} \sum_{m=0}^{+\infty} \left(\frac{C_2}{d^2}\right)^m \left(\frac{C_4}{d^2}\right)^m \left[\frac{1}{\hat{\gamma} - 1}\right]^m \\ & = \frac{2d^2}{C_1} \sum_{m=0}^{+\infty} \left[\frac{C_2 C_4}{d^4(\hat{\gamma} - 1)}\right]^m. \end{aligned} \quad (4.14)$$

For sufficiently large d , $\frac{C_2 C_4}{d^4(\hat{\gamma} - 1)} < 1$ and therefore

$$\tilde{E} \left[\frac{2^{T + \sum_{j=0}^T K_j} M^{6T+4 \sum_{j=0}^T K_j} (1 + \lambda M^2)^{2 \sum_{j=1}^T L_j + 2 \sum_{j=0}^T K_j}}{\lambda^{\sum_{j=1}^T L_j - T} (E\rho^2)^{\sum_{j=1}^T L_j + 2T + 2 \sum_{j=0}^T K_j}} \right] < +\infty \quad (4.15)$$

when $\lambda = \frac{\gamma}{dE\rho^2}$.

Then according to Lemma 4.1,

$$\lambda_c(d) \leq \frac{\gamma}{dE\rho^2}$$

for sufficiently large d and hence

$$\limsup_{d \rightarrow +\infty} d\lambda_c(d) \leq \frac{\gamma}{E\rho^2}$$

for any $\gamma > 1$.

Let γ decrease to 1, then we accomplish the proof. □

Since we have shown that $\liminf_{d \rightarrow +\infty} d\lambda_c(d) \geq \frac{1}{E\rho^2}$ in Section 3, the whole proof of Theorem 2.1 is completed.

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